

# Open-multicommutativity of some functors related to the functor of probability measures

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Abstract

The property of a normal functor to be open-multicommutative proposed by Kozhan and Zarichnyi (2004) is investigated. A number of normal functors related to the functor of probability measures and equipped with convex structure are considered here and it is proved that the functors  $cc$ ,  $ccP$ ,  $G_{cc}P$  and  $\lambda_{cc}P$  are open-multicommutative.

## 1. Introduction

The classical object in topology and functional analysis - spaces of probability measures - is widely used in economics and game theory last ten years (see Lucas (1971), and Prescott (1971), Mas-Colell (1984), Jovanović and Rosenthal (1988)). All this investigations deal with the notion of the set-valued correspondence map which assigns to every probability measures on the factors of the product of compacta the set of probability measures with these marginals,

$$\psi(\mu_1, \dots, \mu_n) = \{\lambda \in P(X_1 \times \dots \times X_n) : P\pi_i(\lambda) = \mu_i, i = 1, \dots, n\}.$$

The problem of continuity of this map can be equivalently redefined in terms of openness of the characteristic map of the bicommutative diagram

$$\begin{array}{ccc} P(X \times Y) & \xrightarrow{P\pi_1} & P(X) \\ \downarrow P\pi_2 & & \downarrow P1_* \\ P(Y) & \xrightarrow{P1_*} & P(\{*\}) \end{array}$$

Due to the well known theorem of Ditor and Eifler (1972) the openness property of the characteristic map is closely related to the property of the bicommutativity of a normal functor and, in particular, the functor of probability measures.

A common generalization of these two properties generates a new notion of multi-commutativity of the normal function which was proposed by Kozhan and Zarichnyi (2004). In this paper they investigate the functor of probability measures and show that this functor is multi-commutative.

In the economic theory, together with the space of probability measures, different convex structures are commonly used, in particular, the spaces of convex closed subsets of the space of probability measures have very nice application.

Here the open-multicommutativity property of normal functors is investigated for functors  $cc$ ,  $ccP$ ,  $G_{cc}P$  and  $\lambda_{cc}P$ .

## 2. Notations and definitions

Let  $\mathbf{K}$  be a finite category. Denote by  $|\mathbf{K}|$  the class of all objects of the category  $\mathbf{K}$ . For every  $A, B \in |\mathbf{K}|$  the set  $\mathbf{K}(A, B)$  consists of all morphisms from  $A$  to  $B$  in  $\mathbf{K}$ . A functor  $D: \mathbf{K} \rightarrow \mathbf{Comp}$  is called a *diagram*.

**Definition 1.** The set of morphisms

$$\left( X \xrightarrow{g_A} D(A) \right)_{A \in |\mathbf{K}|} \quad (1)$$

is said to be a *cone* over the diagram  $D$  if and only if for every objects  $A, B \in |\mathbf{K}|$  and for every morphism  $\varphi: A \rightarrow B$  in  $\mathbf{K}$  the diagram

$$\begin{array}{ccc} & X & \\ g_A \swarrow & & \searrow g_B \\ D(A) & \xrightarrow{D(\varphi)} & D(B) \end{array}$$

is commutative.

**Definition 2.** Cone (1) is called a *limit* of the diagram  $D$  if the following condition is satisfied: for each cone  $C' = \left( X' \xrightarrow{g'_A} D(A) \right)_{A \in |\mathbf{K}|}$  there exists a unique morphism  $\chi_{C'}: X' \rightarrow X$  such that  $g'_A = g_A \circ \chi_{C'}$  for every  $A \in |\mathbf{K}|$ .

Further, we denote this cone by  $\lim(D)$ . The map  $\chi_{C'}$  is called the *characteristic map* of  $C'$ .

**Definition 3.** The cone  $C' = \left( X' \xrightarrow{g'_A} D(A) \right)_{A \in |\mathbf{K}|}$  is called *open-multicommutative* if the characteristic map  $\chi_{C'}$  is open and surjective.

Let  $\mathbf{F}$  be a normal functor in the category **Comp**. Define the diagram  $\mathbf{F}(D) : \mathbf{K} \rightarrow \mathbf{Comp}$  in the following way: for every  $A \in \mathbf{K}$  let  $\mathbf{F}(D)(A) = \mathbf{F}(D(A))$  and for every morphism  $\varphi \in \mathbf{K}(A, B)_{A, B \in |\mathbf{K}|}$  we put  $\mathbf{F}(D)(\varphi) = \mathbf{F}(D(\varphi))$ .

**Definition 4.** The normal functor  $\mathbf{F}$  is called *open-multicommutative* if it preserves open-multicommutative cones, i.e. the cone

$$\mathbf{F}(C') = \left( \mathbf{F}(X') \xrightarrow{\mathbf{F}g'_A} \mathbf{F}(D(A)) \right)_{A \in |\mathbf{K}|}$$

over the diagram  $\mathbf{F}(D)$  is open-multicommutative.

For the normal functor  $\mathbf{F}$  we assume that the cone  $\left( Y_{\mathbf{F}} \xrightarrow{\pi_A} \mathbf{F}(D(A)) \right)_{A \in |\mathbf{K}|}$  is a limit of the diagram  $\mathbf{F}(D)$ .

Let us assume that  $X$  is a compact subspace of some locally convex space  $E$ . We consider a functor  $cc : \mathbf{Conv} \rightarrow \mathbf{Comp}$ . It is defined as

$$cc(X) = \{A \subset X : A \text{ is closed and convex}\} \subset \exp(X).$$

Recall the constructions of well-known inclusion hyperspace and superextension functors. We set

$$G(X) = \{\mathbf{A} \in \exp^2(X) : A \in \mathbf{A} \text{ and } A \subset B \in \exp(X) \Rightarrow B \in \mathbf{A}\}.$$

$$\lambda(X) = \{\mathbf{A} \in G(X) : \mathbf{A} \text{ is a maximal linked system}\}.$$

For properties of the functors  $G$  and  $\lambda$  reader is referred to Teleiko and Zarichnyi (1999).

In order to combine these constructions with convex structure we use the defined above functors and the functor  $cc$ . In such a way we can define a space

$$G_{cc}(X) = \{\mathbf{A} \subset cc(X) : \mathbf{A} \text{ is closed and } A \in \mathbf{A} \text{ and } A \subset B \in cc(X) \Rightarrow B \in \mathbf{A}\}.$$

It is obviously that  $G_{cc}(X) \subset G(X)$  and we can endow it with the topology induced from  $G(X)$ . For a map  $f : X \rightarrow Y$  in **Comp** and  $\mathbf{A} \in G_{cc}(X)$  let

$$G_{cc}f(\mathbf{A}) = \{B \in G_{cc}(Y) \mid f(A) \subset B, A \in \mathbf{A}\}.$$

Thus we can define a covariant functor  $G_{cc} : \mathbf{Conv} \rightarrow \mathbf{Comp}$ . In the same way we can define a functor  $\lambda_{cc} : \mathbf{Conv} \rightarrow \mathbf{Comp}$ .

Due to the natural convex structure of the functor  $P$  we can compose the functors  $G_{cc}$  and  $\lambda_{cc}$  with the functor of probability measures and it gives us two new constructions of functors in the category **Comp**:

$$G_{cc}P = G_{cc} \circ P: \mathbf{Comp} \rightarrow \mathbf{Comp}$$

$$\lambda_{cc}P = \lambda_{cc} \circ P: \mathbf{Comp} \rightarrow \mathbf{Comp}.$$

### 3. Open-multicommutativity of the functor $ccP$ and related functors

**Proposition 1.** *A normal bicommutative functor  $\mathbf{F}$  is multicommutative.*

**Proof.** This Proposition is actually proved in Kozhan and Zarichnyi (2004) for the functor of probability measures. However, in the proof they use only normality and bicommutativity of the functor  $P$  and that is why it can be applied for every normal bicommutative functor.

Q.E.D.

The following proposition is also the generalization of the result of Kozhan and Zarichnyi.

**Proposition 2.** *A normal open functor  $\mathbf{F}$  is open-multicommutative if and only if the characteristic map*

$$\chi_{\mathbf{F}}: \mathbf{F}(X) \rightarrow Y_{\mathbf{F}}$$

*is open for every diagram  $D$  with finite spaces  $D(A)$ ,  $A \in |\mathbf{K}|$ .*

**Proof.** The proof is analogical to the proof of the theorem 1 in Kozhan and Zarichnyi (2004). The special properties of functor  $P$  are used only in the case of finite spaces  $D(A)$ . The rest of the proof consider only general properties of normal bicommutative functors and this implies that the scheme of the proof is the same.

Q.E.D.

**Proposition 3.** *Functors  $\exp$ ,  $G$  and  $\lambda$  are open-multicommutative.*

**Proof.** Let us check whether the conditions of the proposition 2 are satisfied. The functor  $\exp$  is normal and bicommutative (see Teleiko and Zarichnyi (1999)). Assume that every compactum  $D(A)$  is finite for each  $A \in |\mathbf{K}|$ . Since every finite

compactum  $D(A)$  is discrete space therefore  $\exp(D(A))$  is also discrete, which follows from the properties of the Vietoris' topology. It is known (see Kozhan and Zarichnyi (2004)) that  $Y_{\exp} \subseteq \prod_{A \in |\mathbf{K}|} \exp(D(A))$ , which is discrete as well as  $\exp(X)$ , is a subset of the discrete space  $\exp\left(\prod_{A \in |\mathbf{K}|} D(A)\right)$ . Thus the characteristic map  $\chi_{\exp}$  is the map of two discrete spaces and this necessarily implies that it is open. The spaces  $G(X)$  and  $\lambda(X) \subset \exp^2(X)$  for every compactum  $X$  and therefore are also discrete if  $X$  is so. Then normality and bicommutativity of them (see Proposition 2) implies the open-multicommutativity of these functors.

Q.E.D.

**Lemma 1.** *Let  $B \subset T \times T$  and  $\pi_1(B) \subset C \subset T$ . If  $C$  is convex set then this implies that  $\pi_1(\text{conv}(B)) \subset C$ . If in addition we have that  $\pi_1(B) \supset C$  then  $\pi_1(\text{conv}(B)) = C$ .*

**Proof.** Consider an arbitrary point  $x \in \text{conv}(B) \setminus B$ . There exist two points  $x_1, x_2 \in B$  and  $\alpha \in (0, 1)$  such that  $x = \alpha x_1 + (1 - \alpha) x_2$ . Since  $\pi_1(x_1), \pi_1(x_2) \in C$  and  $C$  is convex,  $\pi_1(x) \in C$ . The point  $x$  is arbitrary chosen thus the first statement of the lemma is proved.

The second statement of the lemma is evident.

Q.E.D.

**Proposition 4.** *The functor  $cc: \mathbf{Conv} \rightarrow \mathbf{Comp}$  is open-multicommutative.*

**Proof.** Let us prove that the characteristic map  $\chi_{cc}: cc(X) \rightarrow Y_{cc}$  is open. Consider an arbitrary point  $B \in cc(X)$  and an arbitrary sequence  $\{C_i\}_{i \in \mathbb{N}} \subset Y_{cc}$  such that  $\lim C_i = C$ , where  $C = \chi_{cc}(B)$ . Recall that for every compactum  $X$  with convex structure the space  $cc(X) \subset \exp(X)$ . Since  $B \in \exp(X)$  and  $\chi_{\exp}$  is open map this implies that there exists a sequence  $\{B_i\}_{i \in \mathbb{N}} \subset \exp(X)$  such that

$$\lim B_i = B \text{ and } \chi_{\exp}(B_i) = C_i$$

for every  $i \in \mathbb{N}$ . Denote by  $D_i$  the convex hull of the set  $B_i$ . Since the function  $\text{conv}: \exp(T) \rightarrow cc(T)$  is continuous in the Vietoris topology for every compactum  $T$  with the convex structure, we see that

$$\lim D_i = \lim \text{conv}(B_i) = \text{conv}(\lim B_i) = \text{conv}(B) = B \in cc(X).$$

Each  $C_i$  is a convex set therefore  $\pi_A(C_i)$  is also a convex set for every  $A \in |\mathbf{K}|$  hence Lemma 1 implies that  $\chi_{cc}(D_i) = C_i$  for every  $i \in \aleph$ . Thus, the map  $\chi_{cc}$  is open. The surjectivity of the characteristic map is obvious since every element in  $\prod_{A \in |\mathbf{K}|} cc(D(A))$  is also in  $cc\left(\prod_{A \in |\mathbf{K}|} D(A)\right)$ .

Q.E.D.

**Proposition 5.** *Let categories  $Q_1, Q_2, Q_3 \subset \mathbf{Top}$  and functors  $F_1: Q_1 \rightarrow Q_2$  and  $F_2: Q_3 \rightarrow Q_1$  are open-multicommutative then the composition  $F_1 \circ F_2: Q_3 \rightarrow Q_2$  is also open-multicommutative.*

**Proof.** The open-multicommutativity of the functor  $F_2$  implies that the characteristic map  $\chi_{F_2, D}: F_2(X) \rightarrow Y_{F_2}$  is open and surjective. Since the functor  $F_1$  is open hence the composition  $F_1 \chi_{F_2, D}: F_1 \circ F_2(X) \rightarrow F_1(Y_{F_2})$  is also open and surjective. Consider now a diagram  $D_1$  in the category  $\mathbf{K}$  such that for every  $A \in |\mathbf{K}|$  we have  $D_1(A) = F_2(D(A))$  and for each  $\varphi \in \mathbf{K}(A, B)_{A, B \in |\mathbf{K}|}$  we have  $D_1(\varphi) = F_2(D_1(\varphi))$ . The functor  $F_1$  is open-multicommutative, so this implies that the characteristic map  $\chi_{F_1, D_1}: F_1(Y_{F_2}) \rightarrow Y_{F_1 \circ F_2}$  is open and surjective. Thus, a map  $\chi_{F_1 \circ F_2, D} = F_1 \chi_{F_2, D} \circ \chi_{F_1, D_1}: F_1 \circ F_2(X) \rightarrow Y_{F_1 \circ F_2}$  is open and surjective for any diagram  $D$  as the composition of two open and surjective maps. This implies that the functor  $F_1 \circ F_2$  is open-multicommutative.

Q.E.D.

**Corollary 1.** *The functor  $ccP: \mathbf{Comp} \rightarrow \mathbf{Comp}$  is open-multicommutative.*

**Proof.** This follows from the open-multicommutativity of the functors  $P$  (see Kozhan and Zarichnyi (2004)),  $cc$  (see Proposition 4) and Proposition 5.

Q.E.D.

**Proposition 6.** *The functors  $G_{cc}$  and  $\lambda_{cc}$  are open-multicommutative.*

**Proof.** Let us define a retraction for every

$$r_{cc}X: G(X) \rightarrow G_{cc}(X)$$

in the following way: for every  $\mathbf{A} \in G(X)$

$$r_{cc}X(\mathbf{A}) = \{B \in \exp(X) \mid B = \text{conv}(A), A \in \mathbf{A}\}.$$

It is easy to see that  $r_{cc}X(\mathbf{A}) \in G_{cc}(X)$ .

The base of the space  $G(X)$  is formed by the sets  $(U_1^+ \cap \dots \cap U_m^+) \cap (V_1^- \cap \dots \cap V_n^-)$ , where

$$U^+ = \{\mathbf{A} \in G(X) \mid \text{there exists } A \in \mathbf{A} \text{ such that } A \subset U\}$$

$$U^- = \{\mathbf{A} \in G(X) \mid A \cap U \neq \emptyset \text{ for every } A \in \mathbf{A}\}$$

for some open set  $U \subset X$ .

Let us show that  $r_{cc}X$  is continuous for every  $X \in \mathbf{Conv}$ . To prove the continuity at a point  $\mathbf{A}_0 \in G(X)$  it is sufficient to show that for every element  $U^+ \cap G_{cc}(X)$  (respectively  $U^- \cap G_{cc}(X)$ ) which contains  $\mathbf{A}_0$  there exists a neighborhood  $\mathbf{V} \subset G(X)$  of the point  $\mathbf{B}_0 = r_{cc}X(\mathbf{A}_0)$  such that

$$r_{cc}X(\mathbf{V}) \subset U^+ \cap G_{cc}(X) \text{ ( } U^- \cap G_{cc}(X) \text{ respectively) .}$$

Assume first that  $\mathbf{U} = U^- \cap G_{cc}(X)$ . Denote  $\mathbf{V} = U^-$ . Then for every  $\mathbf{A} \in \mathbf{V}$  we have

$$\forall A \in \mathbf{A}: A \cap U \neq \emptyset \Rightarrow \text{conv}(A) \cap U \neq \emptyset.$$

Since

$$\mathbf{B} = r_{cc}X(\mathbf{A}) = \{B \in \exp(X) \mid B = \text{conv}(A), A \in \mathbf{A}\},$$

this implies that  $\forall B \in \mathbf{B}$  we have  $B \cap U \neq \emptyset$  and then  $\mathbf{B} \in U^- \cap G_{cc}(X) = \mathbf{U}$ .

Assume now that  $\mathbf{U} = U^+ \cap G_{cc}(X)$ . Since  $X$  is a subset of locally convex space  $E$ , there exists a base of  $X$  consisting of convex sets. This implies that we can find an open convex set  $V \subset U$ . Denote  $\mathbf{V} = V^+$ . For every  $\mathbf{A} \in \mathbf{V}$  there is a set  $A_1 \in \mathbf{A}$  such that  $A_1 \subset V$ . The set  $V$  is convex, thus  $\text{conv}(A_1) \subset V \subset U$ . This implies that for  $\mathbf{B} = r_{cc}X(\mathbf{A})$  we can find  $B_1 \in \mathbf{B}$  such that  $B_1 \subset U$  and this means that  $\mathbf{B} \in U^+ \cap G_{cc}(X) = \mathbf{U}$ . Thus, the map  $r_{cc}$  is continuous.

Let us prove that the characteristic map  $\chi_{G_{cc}} : G_{cc}(X) \rightarrow Y_{G_{cc}}$  is open. Consider an arbitrary point  $\mathbf{B} \in G_{cc}(X)$  and an arbitrary net  $\{\mathbf{C}_i\}_{i \in \mathbb{N}} \subset Y_{G_{cc}}$  such that  $\lim \mathbf{C}_i = \mathbf{C}$ , where  $\mathbf{C} = \chi_{G_{cc}}(B)$ . For every compactum  $X$  the space  $G_{cc}(X) \subset G(X)$  and then  $\mathbf{B} \in G(X)$ . Since  $\chi_G$  is open map this implies that there exists a net  $\{\mathbf{B}_i\}_{i \in \mathbb{N}} \subset G(X)$  such that

$$\lim \mathbf{B}_i = \mathbf{B} \text{ and } \chi_G(\mathbf{B}_i) = \mathbf{C}_i$$

for every  $i \in \aleph$ .

Denote by  $\mathbf{D}_i$  an image of the function  $r_{cc}$  of the set  $\mathbf{B}_i$ . Since the function  $r_{cc}$  is continuous, we see that

$$\lim \mathbf{D}_i = \lim r_{cc}(\mathbf{B}_i) = r_{cc}(\lim \mathbf{B}_i) = r_{cc}(\mathbf{B}) = \mathbf{B} \in G_{cc}(X).$$

Each  $\mathbf{C}_i$  is in  $\prod_{A \in |\mathbf{K}|} G_{cc}(D(A))$  and let  $\mathbf{C}'_i = \chi_{G_{cc}}(\mathbf{D}_i)$  for every  $i \in \aleph$ . Let us prove that  $\mathbf{C}_i = \mathbf{C}'_i$ . For every  $A \in |\mathbf{K}|$  we see that

$$G_{cc}\pi_A(\mathbf{D}_i) = \pi_A(\mathbf{C}'_i) = \{C' \in cc(X) \mid \pi_A(D) \subset C', D \in \mathbf{D}_i\}.$$

Lemma 1 proves that if for some  $C \in cc(X)$  the statement  $\pi_A(B) \subset C$  is satisfied for  $B \in \mathbf{B}_i$  then  $\pi_A(D) = \pi_A(\text{conv}(B)) \subset C$ . This implies that  $G_{cc}\pi_A(\mathbf{B}_i) \subset G_{cc}\pi_A(\mathbf{D}_i)$ . On the other hand, if  $\pi_A(D) \subset C'$  for some  $D \in \mathbf{D}_i$  then

$$\pi_A(B) \subset \pi_A(\text{conv}(B)) = \pi_A(D) \subset C' \text{ for } B \in \mathbf{B}_i.$$

This means that  $G_{cc}\pi_A(\mathbf{B}_i) \supset G_{cc}\pi_A(\mathbf{D}_i)$ . Since these two inclusions are satisfied for every  $A \in |\mathbf{K}|$ , we have

$$\prod_{A \in |\mathbf{K}|} G_{cc}\pi_A(\mathbf{B}_i) = \prod_{A \in |\mathbf{K}|} G_{cc}\pi_A(\mathbf{D}_i)$$

and it is equivalent to  $\mathbf{C}_i = \mathbf{C}'_i$ . This immediately implies that  $\chi_{G_{cc}}(\mathbf{D}_i) = \mathbf{C}_i$  for every  $i \in \aleph$ . Thus, the characteristic map  $\chi_{G_{cc}}$  is open and the functor  $G_{cc}$  is open-multicommutative. This map is also surjective since for every  $\mathbf{C} \in Y_{G_{cc}}$  we have  $\chi_{G_{cc}}(\mathbf{D}_C) = \mathbf{C}$ , where

$$\mathbf{D}_C = \{D \in cc(X) \mid D \supset \prod_{A \in |\mathbf{K}|} C_A, C_A \in \pi_A(\mathbf{C})\}.$$

Let us note that the restriction

$$r_{cc}|_{\lambda(X)}: \lambda(X) \rightarrow \lambda_{cc}(X)$$

is also continuous retraction. Using this and the open-commutativity of the functor  $\lambda$  we can in the same way conclude that the functor  $\lambda_{cc}$  is also open-multicommutative.

Q.E.D.

Let us consider now the functors  $G_{cc}P$  and  $\lambda_{cc}P$  which are defined in Teleiko and Zarichnyi (1999). Actually,  $G_{cc}P(X) = G_{cc}(P(X))$  for every compactum  $X$



and it is the composition of two functors  $G_{cc}$  and  $P$  (the same situation is for the functor  $\lambda_{cc}P$ ). The following result can be derived from Propositions 4 and 6.

**Corollary 2.** *The functors  $G_{cc}P$  and  $\lambda_{cc}P$  are open-multicommutative.*

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